BASKET, FLAT PLUMBING AND FLAT PLUMBING BASKET NUMBERS OF LINKS

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Abstract. We define the basket number, the flat plumbing number and the flat plumbing basket number of a link. Then we provide some upperbounds for these plumbing numbers by using Seifert's algorithm. We study the relation between these plumbing numbers and the genera of links.

1. Introduction

Let L be a link in $\mathbb{S}^3$. A compact orientable surface $\mathcal{F}$ is a *Seifert surface* of L if the boundary of $\mathcal{F}$ is L. The existence of such a surface was first proven by Seifert using an algorithm on a diagram of L, named after him as *Seifert's algorithm* [25]. There are many interesting results about Seifert surfaces. One of classical link invariants, the *genus* of a link L can be defined by the minimal genus among all Seifert surfaces of L, denoted by $g(L)$.

Some of Seifert surfaces of links feature extra structures. Seifert surfaces obtained by plumbings annuli are the main focus of this article. Even though a higher dimensional plumbing can be defined but here we will only concentrate on 1-sphere plumbing as shown in Figure 1. It is often called a Murasugi sum and it has been studied extensively for the fibreness of links and surfaces [8–10, 12, 16, 18, 21, 26]. Rudolph has introduced several plumbed Seifert surfaces [23]. To show the existence of these Seifert surfaces, it is common to present the link as the closure of a braid in classical Artin group [7, 13]. Furthermore, a few different ways to find its braid presentations were found by Alexander [2], Morton [17], Vogel [28] and Yamada [29]. In particular, the work of Yamada is closely related with Seifert's algorithm and has been generalized to find another wonderful presentation of the braids groups [3]. We will use Seifert's algorithm to show the existence of basket surfaces, flat plumbing surfaces and flat plumbing basket surfaces [7, 13]. Consequently we can define the basket number, the flat plumbing number and the flat plumbing basket number of a link. Then we also study some upperbounds for these plumbing numbers and the relations between these plumbing numbers and several genera. We refer to [1] for the standard terms of knot theory.

The outline of this paper is as follows. In section 2, we review definitions of plumbing numbers. Then we find some upperbounds for these plumbing numbers. In section 3 we study the relations between these plumbing numbers and several genera of a link.

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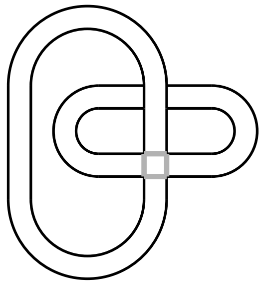

FIGURE 1. An example of 1-sphere plumbing.

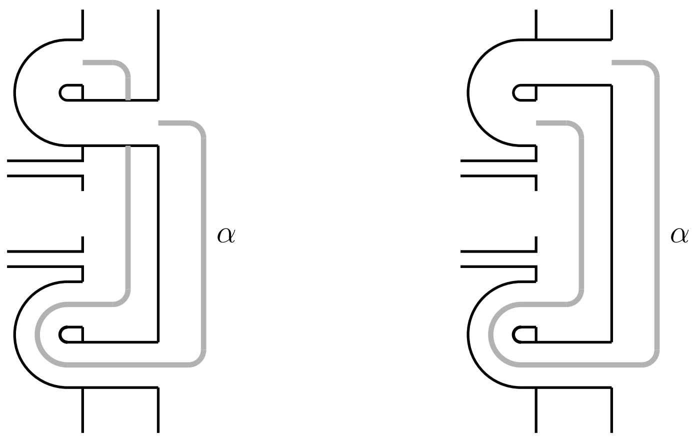

FIGURE 2. Deplumbing A_2 and A_0 annulus on $\mathcal{F}$ along α.

2. Plumbing numbers

There are several interesting plumbing numbers [20–23] but we would like to discuss three plumbing numbers. We will review the definitions of these plumbing numbers and prove theorems on the upperbound of each plumbing number. For standard definitions and notations, we refer to [23]. Throughout the section, we will assume all links are not splitable, *i.e.*, Seifert surfaces are connected. Otherwise, we can handle each connected component separately.

2.1. **Basket number.** Let $A_n \subset \mathbb{S}^3$ denote an n-twisted unknotted annulus. A Seifert surface $\mathcal{F}$ is a *basket surface* if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_0 *_\alpha A_n$ which can be constructed by plumbing A_n to a basket $\mathcal{F}_0$ along a proper arc $\alpha \subset D_2 \subset \mathcal{F}_0$ [23]. First, we define a *basket number* of a link L, denoted by $bk(L)$, to be the minimal number of annuli used to obtain a basket surface $\mathcal{F}$ such that $\partial\mathcal{F} = L$.

If the link L is presented as a closed braid, we can find the following upperbound for the basket number of L. By a simple modification of the idea of main theorems in [7, Theorem 2.4] [13, Theorem], we have the following theorem.

Theorem 2.1. *Let L be a link which is the closure of a braid $\beta \in B_n$ where the braid β can be written as $\sigma_{n-1}\sigma_{n-2}\ldots\sigma_2\sigma_1 W$ and the length of the word W is m. Then the basket number of L is less than or equal to m, i.e., $bk(L) \leq m$.*

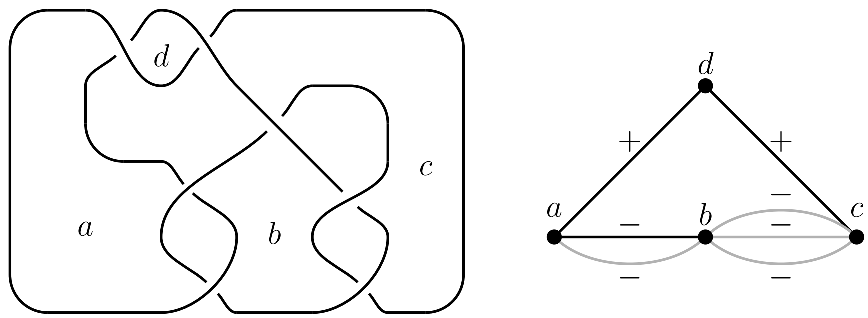

FIGURE 3. A link and its corresponding induced graph with a spanning tree.

Proof. First we pick a disc D which is obtained from n disjoint disks by attaching $(n-1)$ twisted bands presented by $\sigma_{n-1}\sigma_{n-2}\dots\sigma_2\sigma_1$. For each letter in the word W, it is fairly easy to see that we can remove the crossing presented by the given letter using either one of deplumbings along arc α as depicted in Figure 2, *i.e.*, the length of the resulting word is now $m-1$. Inductively, we can see that $\mathcal{F}$ can be obtained from the disc D by plumbing at most m times. □

Next, we find another upperbound by using Seifert's algorithm. Once we have a Seifert surface $\mathcal{F}$ of a link L by applying Seifert's algorithm to a link diagram S as shown in Figure 3, we construct a (signed) graph $G(L)$ by collapsing discs to vertices and half twist bands to signed edges as illustrated in the right side of Figure 3. Let us call it an *induced graph* of a link L. It is fairly easy to see that the number of Seifert circles (half twisted bands), denoted by $s(S)(c(S))$, is the cardinality of the vertex set (edge set, respectively). It is clear that if Seifert surface $\mathcal{F}$ is connected, its induced graph $G(L)$ is also connected. For terms in graph theory, we refer to [11]. Now we are set to state a theorem for a new upperbound.

Theorem 2.2. *Let $\mathcal{F}$ be a canonical Seifert surface of a link diagram S of L with $s(S)$ Seifert circles and $c(S)$ half twisted bands. Let T be a spanning tree of its induced graph $G(L)$ which has exactly $s(S)-1$ edges. Then the basket number of L, $bk(L) \leq c(S)-s(S)+1$.*

Proof. Let D be the disc corresponding to the spanning tree T. For any twisted band $t(e)$ which is not a part of the disc D, *i.e.*, an edge e in $E(G)-T$ in its induced graph we described above, we can choose a unique path in T which joins the ends of the edge. Using this path, we can find an arc α such that $\mathcal{F} = \mathcal{F}_0 *_\alpha A_n$ which can be constructed by plumbing A_n to a basket $\mathcal{F}_0$ along a proper arc $\alpha \subset D \subset \mathcal{F}_0$ where $\mathcal{F}_0$ is the surface obtained from $\mathcal{F}$ by removing the twisted band $t(e)$. Inductively, we can show that the Seifert surface $\mathcal{F}$ is a basket and its basket number is less than or equal to $c(S) - s(S) + 1$. □

2.2. **Flat plumbing number.** A Seifert surface $\mathcal{F}$ is a *flat plumbing surface* if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_0 *_\alpha A_0$ which can be constructed by plumbing A_0 to a flat plumbing surface $\mathcal{F}_0$ along a proper arc $\alpha \subset \mathcal{F}_0$. Remark that the gluing regions in the construction are not necessarily contained in D_2. Hayashi and Wada showed every oriented links in $\mathbb{S}^3$ bounds a flat plumbing surface by finitely many flat plumbings [13]. Thus, we can define the *flat*

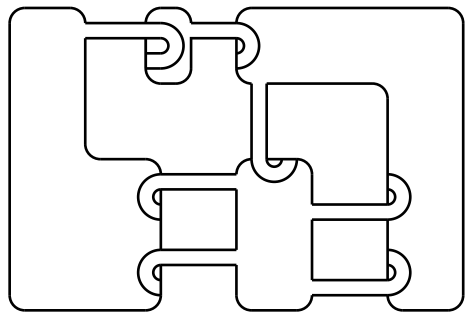

FIGURE 4. A flat Seifert surface of the link in Figure 3.

plumbing number of L, denoted by $fp(L)$, to be the minimal number of flat annuli to obtain a flat plumbing surface. By using the original work of [13], one can find an upperbound from the braid word of the link L in the following theorem.

Theorem 2.3. *([13]) Let L be an oriented link which is a closed n-braid with a braid word W where the length of W is m, then there exists a flat Seifert surface F with $m+n-1$ bands such that ∂F is isotopic to L, i.e., $fp(L) \leq m+n-1$.*

Next, we find an upperbound for the flat plumbing number by using Seifert's algorithm.

Theorem 2.4. *Let L be a link with a canonical Seifert surface $\mathcal{F}$ of a diagram S with $s(S)$ Seifert circles and $c(S)$ half twisted bands. Let T be a spanning tree of $G(L)$ which has exactly $s(S)-1$ edges. Then the flat plumbing number of L is less than or equal to $c(S)+s(S)-1$, i.e., $fp(L) \leq c(S)+s(S)-1$.*

Proof. Let D be the disc corresponding to the spanning tree T. If we want to use the same idea of Theorem 2.2, then there is a possibility that the plumbing annulus may not be flat since the neighborhood of α could be twisted as many as the sum of signs of the edges in the path in $G(L)$ that α has passed. However, we have a flexibility that the arc α can pass the annulus we had plumbed. For each edge in the spanning tree T, we are consecutively plumbing two annuli which have different signs. The effect of this does not change the link type because it is exactly a type II Reidermeister move. Every adjacent vertices in the spanning tree T are now connected by three edges, one from the original spanning tree T and two from new edges of sign $+$ and $-$, let us denote it by T'. For any twisted band $t(e)$ of sign ϵ which is not a part of the disc D, *i.e.*, corresponding to an edge e in $E(G)-T$, we can find a path P in T' which joins the ends of the edge and the sum of signs of the edges in the path P is $-\epsilon$ because the surface is oriented, *i.e.*, the length of this path is odd integer. We can find an arc α which is corresponding to the path P such that $\mathcal{F} = \mathcal{F}_0 *_\alpha A_0$ which can be constructed by plumbing A_0 to a flat plumbing surface $\mathcal{F}_0$ along a proper arc $\alpha \subset \mathcal{F}_0$ where $\mathcal{F}_0$ is the surface obtained from $\mathcal{F}$ by removing the twisted band $t(e)$. Inductively, we find a Seifert surface $\mathcal{F}$ which is a flat plumbing surface and whose flat plumbing number is less than or equal to $c(S)+s(S)-1$. ☐

In general, it is not necessary to plumb two annuli for each edge in a spanning tree T to have the property that for any twisted band $t(e)$ of sign ϵ which is not a part of the disc D, we can find a path in T' which joins the ends of the edge and where the sum of signs

in the path is $-\epsilon$. For example, one can see that for the link L depicted in Figure 3, we only need to plumb two annuli between vertices a and b to have a spanning tree T' of the desired property. Therefore, the flat plumbing number of L in Figure 3 is less than 7.

2.3. **Flat plumbing basket number.** A Seifert surface $\mathcal{F}$ is a *flat plumbing basket surface* if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_0 *_\alpha A_0$ which can be constructed by plumbing A_0 to a basket $\mathcal{F}_0$ along a proper arc $\alpha \subset D_2 \subset \mathcal{F}_0$. We say that a link L admits a *flat plumbing basket presentation* if there exists a flat plumbing basket $\mathcal{F}$ such that $\partial\mathcal{F}$ is equivalent to L. In [7], it is shown that every link admits a flat plumbing basket presentation. So we can define the *flat plumbing basket number* of L, denoted by $fpbk(L)$, to be the minimal number of flat annuli to obtain a flat plumbing basket surface of L. By presenting a link by a closed braid, we have the following theorem.

Theorem 2.5. *([7]) Let L be an oriented link which is a closed n-braid with a braid word $\sigma_{n-1}\sigma_{n-2}\dots\sigma_1 W$ where the length of W is m and W has s positive letters, then there exists a flat plumbing basket surface $\mathcal{F}$ with $m+2s$ bands such that $\partial\mathcal{F}$ is isotopic to L, i.e., $fpbk(L) \leq m+2s$.*

By combining the ideas of Theorem 2.4, we can improve the upperbound in Theorem 2.5 as follows. Let L be an oriented link which is a closed n-braid with a braid word W. Since we can add $\sigma_i\sigma_i^{-1}$, without loss of the generality, we assume σ_i and σ_i^{-1} appear in W, $i = 1, 2, \dots, n-1$. Let $a_i(1)$ and $a_i(-1)$ be the number of σ_i and σ_i^{-1} respectively in W. Let $\epsilon_i = -\dfrac{a_i(1) - a_i(-1)}{|a_i(1) - a_i(-1)|}$ for $i = 1, 2, \dots, n-1$. Then, we choose a disc which is presented by $\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_3}\dots\sigma_{n-1}^{\epsilon_{n-1}}$. The main idea of Theorem 2.5 is that each σ_i in W can be replaced by σ_i^{-1} by plumbing two flat annuli. Since we replace each $\sigma_i^{\epsilon_i}$ in W by $\sigma_i^{-\epsilon_i}$ by plumbing two flat annuli, we have the following theorem.

Theorem 2.6. *Let L be an oriented link which is a closed n-braid with a braid word W which contains σ_i and σ_i^{-1}, $i = 1, 2, \dots, n-1$. Let a_i, ϵ_i be the numbers described above. Then the flat plumbing basket number of L is bounded by $\sum_{i=1}^{n-1} a_i(-\epsilon_i) + 2(a_i(\epsilon_i) - 1)$, i.e.,*

$$fpbk(L) \leq \sum_{i=1}^{n-1} a_i(-\epsilon_i) + 2(a_i(\epsilon_i) - 1).$$

Now we want to find an upperbound by using Seifert's algorithm. For a flat plumbing surface, we can pick an arc α on the flat plumbing surface obtained by plumbing along each edge of a spanning tree of $G(L)$. But for a flat plumbing basket surface, α has to stay in the disk D which was fixed from the beginning, thus we have to choose a disk D carefully. The spanning tree of the induced graph from a closed braid is a path itself. Thus, there is no ambiguity about the choice of a spanning tree for a closed braid. But in general, it could be drastically changed. Surprisingly, even if we choose any spanning tree, an alternative label on the tree with respect to the depth of the tree satisfies the desired property as shown in the following lemma.

Lemma 2.7. *Let G be the induced graph of a Seifert surface. Let T be a spanning tree with an alternating signing with respect to the depth of the tree. Let e be an edge in $E(G) - T$, the sign sum of the path in T which joins the end points of e is either 1 or -1.*

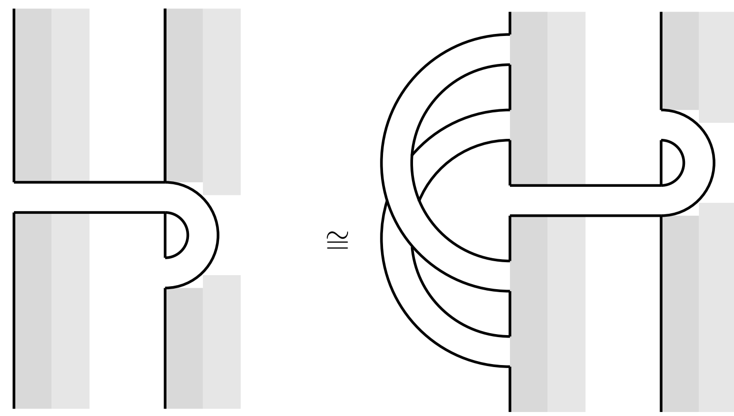

FIGURE 5. How to change the sign of a twisted band by plumbing two flat annuli.

Proof. First of all, it is straightforward that the sign sum of the path must be an odd number because the graph is obtained from a Seifert surface which is an oriented surface. Let P be the path in T which joins the endpoints of e. By the definitions of the tree and alternating sign, P is a union of two paths of alternating signs: one has odd length and the other has even length possibly zero. Therefore, any path joining the end points of an edge e in $E(G) - T$ has a sign sum either 1 or -1. $\square$

Using Lemma 2.7, we can consider the following minimum. First if the sign of an edge e in T does not coincide with a sign of the edge between the endpoints of the edge e, then we have to isotope the link by a type II Reidermeister move. Since we can completely reverse the sign of all edges in the spanning tree T, we may assume the total number of type II Reidermeister moves in the process is less than or equal to $\left\lceil \frac{s(S)-1}{2} \right\rceil$. Let us write β to be the total number of type II Reidermeister moves in the process described above. For each edge e in $E(G)-T$, if the sign of the edge is different from the sign sum of the edges in the path P which joins the endpoints of e, then we can plumb a flat annulus along a curve α corresponding to the path P in the spanning tree T. Otherwise, we need to add three flat annuli to make the half twisted band presented by the edge e as shown in Figure 5 [7]. Let γ be the total number of the edges in $E(G) - T$ whose signs are different from the sign sum of the edges in the paths which join the endpoints of the edges. If we set D the disc corresponding to the spanning tree T, then we get the following theorem which is obvious from the construction.

Theorem 2.8. *Let $\mathcal{F}$ be a canonical Seifert surface of a link L with $s(S)$ Seifert circles and $c(S)$ half twisted bands. Let G be the induced graph of a Seifert surface $\mathcal{F}$. Let T be a spanning tree with an alternating signing with respect to the depth of the tree. Let β and γ be the numbers described above. Then the flat plumbing basket number of L is bounded by $(3c(S) - 2\gamma) - (3s(S) - \beta) + 3$, i.e.,*

$$fpbk(L) \leq (3c(S) - 2\gamma) - (3s(S) - \beta) + 3.$$

We may attain a better upperbound for Theorem 2.8 if we use the method in the proof of Theorem 2.6 by considering all spanning trees with all possible signings on the trees with the property that any path joining the end points of an edge e in $E(G) - T$ has a sign sum either 1 or -1.

3. Relations between plumbing numbers

First we will look at the relations between three plumbing numbers. Fundamental inequalities regarding these three basket numbers are

$$bk(L) \leq fp(L) \leq fpbk(L).$$

For the second inequality, there exists a link that the inequality is proper [7]. For the first inequality, we consider L_{2n}, the closure of $(\sigma_1)^{2n} \in B_2$. For $n \neq 0$, it is a nontrivial link and it can be obtained by plumbing one annulus A_{2n}, so its basket number is 1. On the other hand, it is fairly easy to see that any link whose flat plumbing number is less than 3 is trivial. Thus, the first equality is proper for L_{2n}. Then naturally we can ask the following question.

Question 3.1. *Are there links for which the difference of plumbing numbers is arbitrary large ?*

Next we will relate the plumbing numbers with one of classical link invariants. Let us recall the definitions first. The *genus* of a link L can be defined by the minimal genus among all Seifert surfaces of L, denoted by $g(L)$. A Seifert surface $\mathcal{F}$ of L with the minimal genus $g(L)$ is called a *minimal genus Seifert surface* of L. A Seifert surface of L is said to be *canonical* if it is obtained from a diagram of L by applying Seifert's algorithm. Then the minimal genus among all canonical Seifert surfaces of L is called the *canonical genus* of L, denoted by $g_c(L)$. A Seifert surface $\mathcal{F}$ of L is said to be *free* if the fundamental group of the complement of $\mathcal{F}$, namely, $\pi_1(\mathbb{S}^3 - \mathcal{F})$ is a free group. Then the minimal genus among all free Seifert surfaces of L is called the *free genus* for L, denoted by $g_f(L)$. Since any canonical Seifert surface is free, we have the following inequalities.

$$g(L) \leq g_f(L) \leq g_c(L).$$

There are many interesting results about the above inequalities [4, 6, 14, 15, 18, 24].

For the basket number, we find the following theorem.

Theorem 3.2. *Let L be a link, let l be the number of components of L. Then the basket number of L is bounded as,*

$$2g(L) + l - 1 \leq bk(L) \leq 2g_c(L) + l - 1.$$

Proof. Let V, E and F be the numbers of vertices, edges and faces, respectively in a minimal canonical embedding of $G(L)$. From Theorem 2.2, we find $bk(L) \leq c(S) - s(S) + 1 = E - V + 1 = (E - V - F) + F + 1 = 2g_c(L) - 2 + F + 1 = 2g_c(L) + F - 1$. The first inequality follows from the definition of the genus of a link. □

Consequently we find the following corollary.

Corollary 3.3. *If L is a link of l components and its minimal genus surface of genus $g(L)$ can be obtained by applying Seifert algorithm on a diagram of L, i.e., $g(L) = g_c(L)$, then $bk(L) = 2g(L) + l - 1$.*

In fact, there are many links for which their genera and canonical genera are the same such as alternating links, closures of positive braids. For these links, we can find their basket number by Corollary 3.3.

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